# Proof of the Newtonian Theorem on the Expansion of the Powers of the Binomial for the cases in which the Exponents are not integer numbers *

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**§1** The theorem, which is usually represented this way

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3 + \text{etc.},$$

if extended as far as possible, i.e.,assumed to hold for all possible numbers assumed for the exponent $n$, constitutes the foundation of whole elementary analysis; hence it is necessary to demonstrate its truth most rigorously. But the way, how this theorem was obtained, namely by multiplying the quantity $a+b$ by itself several times, is of such a nature that only positive integer numbers result for the exponent $n$, since by multiplying the quantity $a+b$ by itself several times, only those powers can result whose exponents indicate the number of factors, which can obviously only be an integer number. Nevertheless, there seems to be hardly any doubt that, if this formula was true for all integer numbers assumed for $n$, the same will also be true for completely all numbers, no matter whether they are fractional or even irrational numbers; although this conclusion is true in this case, this is for other reasons, since one

*Original Title: "Demonstratio theorematis Neutoniani de evolutione potestatum binomii pro casibus, quibus exponentes non sunt numeri integri",first published in „*Novi Commentarii academiae scientiarum Petropolitanae 19* 1775, pp. 103-111", reprint in „*Opera Omnia*: Series 1, Volume 15, pp. 207 - 216 ", Eneström-Number E465, translated by: Alexander Aycock for the project „Euler-Kreis Mainz" can exhibit other cases of such a kind, in which a certain formula is detected to be true, if the exponent $n$ was a positive integer, but the same formula can not hold by any means, if fractional values are attributed to the same exponent.

**§2** To illustrate this with an example, let the following series be propounded

$$\frac{1-a^n}{1-a}+\frac{(1-a^n)(1-a^{n-1})}{1-a^2}+\frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{1-a^3}$$
$$+\frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})}{1-a^4}+\text{etc.},$$

whose value, if the exponent $n$ was a positive integer, is always detected to be equal to that exponent, and nevertheless it is not possible to conclude from this that this equality also holds, if one takes other numbers for $n$; this property indeed also holds for $n=0$; for, then, because of $a^n=1$, the first term immediately vanishes together with all the following ones, which have the factor $1-a^n=0$, such that in this case our series becomes $=0$, i.e. equal to the exponent $n=0$; but then, having taken $n=1$, the first term becomes

$$\frac{1-a}{1-a}=1,$$

but the second term, because of $1-a^{n-1}=0$, vanishes together with all the following ones, such that in this case $n=1$ our series becomes $=1$. Let us also consider the case $n=2$, in which the first term becomes

$$\frac{1-a^2}{1-a}=1+a,$$

but the second term yields

$$\frac{(1-a^2)(1-a)}{1-a^2}=1-a,$$

the third term on the other hand vanishes together with all the following ones because of the factor $1-a^{n-2}=0$, whence the sum of our series will be $=2$, i.e. equal to $n$. Let us also set $n=3$ and the first term will give

$$\frac{1-a^3}{1-a}=1+a+a^2,$$

the second term gives

$$\frac{(1-a^3)(1-a^2)}{1-a^2}=1-a^3$$

and the third

$$\frac{(1-a^3)(1-a^2)(1-a)}{1-a^3}=1-a-aa+a^3,$$

the fourth term and all the following ones on the other hand, since they contain the factor $1-a^{n-3}=0$, vanish, whence our series in this case $n=3$ becomes $=3$. And in like manner one can show, whatever integer number is taken for $n$, that our series will be equal to that number; but it is easily seen, if one would take $n=\frac{1}{2}$, that this series would deviate immensely from the value $\frac{1}{2}$.

**§3** Therefore, since this formula

$$n=\frac{1-a^n}{1-a}+\frac{(1-a^n)(1-a^{n-1})}{1-a^2}+\frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{1-a^3}+\text{etc.}$$

is always true, if $n$ was a positive integer number, but this equality does not hold for other numbers, it is not certain, whether our theorem

$$(a+b)^n=a^n+\frac{n}{1}a^{n-1}b+\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3+\text{etc.}$$

is true in general, even though we are certain that it is true, if the exponent $n$ was a positive integer number. Therefore, it is even more necessary to demonstrate its truth rigorously. I certainly once gave a proof derived from the analysis of the infinite; but since this analysis is based on our theorem, I now see that it is to be rejected being based on that source; a proof not based on this was given by Aepinus in Tomo VIII Novor. Commentar., where he, assuming the general series

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\text{etc.}$$

for the formula $(x+1)^n$, applying a most ingenious method derived the values of several of the coefficients $A$, $B$, $C$, $D$ etc. and from their agreement with Newton's series was then able to conclude that also all the remaining ones follow the same rule; nevertheless, that extraordinary proof was based mainly on induction, furthermore it should also be noted that the second coefficient $B$ can not be obtained by that method, but has to be derived from other rather unnatural conditions. Hence I am confident that the Geometers will be grateful for my proof, since in it nothing is based on induction.

**§4** First, we note that, since

$$a + b = a\left(1 + \frac{b}{a}\right),$$

it will also be

$$(a+b)^n = a^n\left(1 + \frac{b}{a}\right)^n$$

and hence the whole task is reduced to the expansion of this power

$$\left(1 + \frac{b}{a}\right)^n,$$

which, by putting $\frac{b}{a} = x$, is further reduced to $(1+x)^n$, which we know, if the exponent $n$ was a positive integer number, to be equal to this series

$$1 + \frac{n}{1}x + \frac{n}{1} \cdot \frac{n-1}{2}x^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}x^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}x^4 + \text{etc.};$$

but, if $n$ was not a positive integer number, let us consider the value of this series as not known and use the sign $[n]$ for it, such that in general we have

$$[n] = 1 + \frac{n}{1}x + \frac{n}{1} \cdot \frac{n-1}{2}x^2 + \text{etc.},$$

about which we now even do not know more than that in the case, in which $n$ is a positive integer number, it will be

$$[n] = (1+x)^n;$$

but let us investigate, which values this sign $[n]$ obtains in the remaining cases; it will become clear that it will also in general be

$$[n] = (1+x)^n,$$

whatever number is taken for the exponent $n$, having demonstrated which we will have proved the theorem.

**§5** For this investigation let us multiply two series of this kind or two such signs $[n]$ and $[m]$ by each other, so that we obtain a series equal to the product $[m] \cdot [n]$, which is easily seen to be expressed by a form of this kind

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.};$$

to find out how its coefficients $A$, $B$, $C$, $D$, $E$ etc. are determined in terms of the two letters $m$ and $n$, let us at least start the multiplication

$$\begin{aligned}
[m] \qquad &= 1 + \frac{m}{1}x + \frac{m}{1} \cdot \frac{m-1}{2}x^2 + \text{etc.},\\
[n] \qquad &= 1 + \frac{n}{1}x + \frac{n}{1} \cdot \frac{n-1}{2}x^2 + \text{etc.},\\
[m] \cdot [n] &= 1 + \frac{m}{1}x + \frac{m}{1} \cdot \frac{m-1}{2}x^2 + \text{etc.},\\
&\quad + \frac{n}{1}x + \frac{m}{1} \cdot \frac{n}{1} x^2 + \text{etc.},\\
&\quad + \frac{n}{1} \cdot \frac{n-1}{2} x^2 + \text{etc.}
\end{aligned}$$

If we now compare this initial product to the assumed form

$$1 + Ax + Bx^2 + Cx^3 + \text{etc.},$$

which we assumed to result for the product, it is immediately understood that it will be

$$A = m + n \quad \text{and} \quad B = \frac{m}{1} \cdot \frac{m-1}{1} + \frac{m}{1} \cdot \frac{n}{1} + \frac{n}{1} \cdot \frac{n-1}{1}$$

or

$$B = \frac{mm - m}{2} + mn + \frac{nn - n}{2},$$

whence

$$B = \frac{m+n}{1} \cdot \frac{m+n-1}{2}.$$

**§6** As it was possible here to determine the first coefficients $A$ and $B$ in terms of the letters $m$ and $n$, so it is manifest, if the above multiplication would be continued, that hence also the following coefficients $C$, $D$, $E$ etc. can be defined in terms of the same letters $m$ and $n$, although the calculation would become cumbersome quickly and would require a lot of work. Nevertheless, we are now able to conclude that all coefficients $A$, $B$, $C$, $D$, $E$ etc. must be determined in terms of the same two letters $m$ and $n$, even though we do not know exactly how; but here it should especially be stressed that this composition does not depend on the nature of the letters $m$ and $n$, but it will be this way, no matter whether these letters $m$ and $n$ denote integer numbers or any other numbers. This unusual way of reasoning should carefully be noted, since our whole proof is based on it.

§7 Hence this provides us with an easy way to find the true values of all coefficients $A$, $B$, $C$, $D$, $E$ etc., of course by considering the letters $m$ and $n$ as integer numbers, since hence the same determinations result as if they would denote any other numbers. But considering the letters $m$ and $n$ as integer numbers we obviously have

$$[m] = (1+x)^m \quad \text{and} \quad [n] = (1+x)^n,$$

whence the product of these formulas will be

$$[m] \cdot [n] = (1+x)^{m+n};$$

but now this power expanded into this series is

$$1 + \frac{m+n}{1}x + \frac{m+n}{1} \cdot \frac{m+n-1}{2}x^2 + \frac{m+n}{1} \cdot \frac{m+n-1}{2} \cdot \frac{m+n-2}{3}x^3 + \text{etc.}$$

Therefore, if we consider the letters $m$ and $n$ in general, this series must be denoted by $[m+n]$, whence we obtain this extraordinary truth that always is

$$[m] \cdot [n] = [m+n],$$

no matter which numbers are used for these letters.

§8 Therefore, since two of the formulas $[m]$ and $[n]$, if multiplied by each other, yield a simple formula of that nature, so also several formulas of this kind, if multiplied by each other, will be reduced to a simple one; namely, we will have the following reductions

$$\begin{aligned} &[m] \cdot [n] = [m+n], \\ &[m] \cdot [n] \cdot [p] = [m+n+p], \\ &[m] \cdot [n] \cdot [p] \cdot [q] = [m+n+p+q] \end{aligned}$$

$$\text{etc.};$$

hence, if all these numbers $m$, $n$, $p$, $q$ etc. are equal, say $= m$, we will obtain the following reductions of the powers

$$[m]^2 = [2m], \quad [m]^3 = [3m], \quad [m]^4 = [4m] \quad \text{etc.}$$

whence in general it will be

$$[m]^a = [am],$$

while $a$ denotes an arbitrary integer number.

**§9** Having noted all this in advance, let $i$ be an arbitrary positive integer number and let us first set $2m = i$ that $m = \frac{i}{2}$, and the first of the last formulas will give

$$\left[\frac{i}{2}\right]^2 = [i];$$

but since $i$ is an integer number, it will be

$$[i] = (1+x)^i$$

(see § 4) and so it will be

$$\left[\frac{i}{2}\right]^2 = (1+x)^i,$$

whence by extracting the square root

$$\left[\frac{i}{2}\right] = (1+x)^{\frac{i}{2}}$$

and this way we already found that the Newtonian theorem is also true in the cases in which the exponent $n$ is a fraction of the form $\frac{i}{2}$.

**§10** If in like manner we put $3m = i$ that $m = \frac{i}{3}$, the above formulas yield

$$\left[\frac{i}{3}\right]^3 = [i] = (1+x)^i;$$

hence by extracting the root we obtain

$$\left[\frac{i}{3}\right] = (1+x)^{\frac{i}{3}}$$

and therefore out theorem is also true, if the exponent $n$ was a fraction of the form $\frac{i}{3}$; and hence it is manifest that in general it will be

$$\left[\frac{i}{a}\right] = (1+x)^{\frac{i}{a}},$$

such that is has now been demonstrated that our theorem is true, if an arbitrary fraction $\frac{i}{a}$ is taken for the exponent $n$, whence the truth has been shown for all positive numbers taken for the exponent $n$.

**§11** Therefore, it only remains to show the truth also for the cases, in which the exponent $n$ is a negative number. For this purpose recall the reduction we found first

$$[m] \cdot [n] = [m+n],$$

where $m$ denotes a positive number, either an integer or a fraction, so that, as we just demonstrated,

$$[m] = (1+x)^m;$$

further, set $n = -m$ and it will be $m + n = 0$ and hence

$$[0] = (1+x)^0 = 1,$$

having substituted which values the above formula yields

$$(1+x)^m \cdot [-m] = 1,$$

whence we conclude

$$[-m] = \frac{1}{(1+x)^m} = (1+x)^{-m};$$

and so it has now been demonstrated that the Newtonian theorem is also true, if the exponent $n$ is an arbitrary negative fraction; and hence this theorem has been confirmed by most rigorous reasoning.